Generalized Lotka-Volterra (GLV) Models

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Abstract

The Generalized Lotka-Volterra (GLV) model:

$$w_i(t+1) = \lambda w_i(t) + a\bar{w}(t) - c\bar{w}(t)w_i(t) \quad , \quad i = 1, ..., N$$

provides a general method to simulate, analyze and understand a wide class of phenomena that are characterized by power-law probability distributions:

$$P(w)dw \sim w^{-1-\alpha}dw \quad (\alpha \geq 1)$$

and truncated Levy flights fluctuations $L_\alpha(\bar{w})$.

I. INTRODUCTION

Many natural and man-made phenomena are known to involve power-law probability distributions (e.g. Pareto 1897; Zipf 1949; Mandelbrot 1961, 1951, 1963; Atkinson and Harrison 1978; Bak et al. 1997; Bouchaud et al. 97; Cahalan and Joseph 1989; Mantegna and Stanley 1994, 1995, 1996, 1997; Stanley et al. 1995; Sornette et.al. 1997; Zanette and Manrubia 1997, Zhang et al 1997).

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Power-laws are common in systems composed of units that have no characteristic size, and in systems made of auto-catalytic elements (Yule 1924; Champernowne 1953; Simon and Bonini 1958; Ijiri and Simon 1977, Anderson 1995). Moreover it has been shown that in systems which are not separable into energetically independent parts, the power laws take naturally the place of the usual exponential (Boltzmann) distribution (Tsallis 1988).

It has been shown both theoretically (Solomon 1998; Solomon and Levy 1996) and using numerical simulations (Biham et al. 1998) that the Generalized Lotka-Volterra (GLV) model produces power-law distributions, but doesn't suffer from the problems and limitations of previous similar models. Since the GLV requires only a few, and not too restrictive preconditions, it may be expected to be widely applicable.

The GLV involves three scalar parameters, and a probability distribution, each having clear roles in the model's interpretation. Certain model properties are "universal" i.e. independent of some of the model's parameters. This conceptual simplicity makes the GLV easily adaptable to different systems, and makes it possible to use it as a descriptive, understanding and comparison tool.

The features listed above make the GLV a candidate for a general method to simulate, analyze and understand a wide class of phenomena which are characterized by power-law probability distributions and multiscale fluctuations.

The aim of this paper is to provide a practical introduction to GLV modeling. We describe the basic theory, how to construct a numerical simulation of the GLV and applications to practical interesting systems.

For didactic reasons we will present a few previous and simpler models before introducing the GLV. This will nicely partition the topics involved into more manageable parts. The models described in this paper are:

- Single-agent random multiplicative process without barrier eq. (7).
- Single-agent multiplicative process with fixed lower barrier eq. (20).
- Multiple agent process with barrier coupled to the mean eq. (40-42).

- Single-agent linear (stochastic multiplicative/additive) process eq. (50)

- Generalized Lotka-Volterra (autocatalytic competing) agents eq. (52).

II. SOME BASIC CONCEPTS

Microscopic representation or agent-oriented simulation is a relatively new way of generating and expressing knowledge about complex systems (see Solomon 1995 for a physicist oriented review and Kim and Markowitz 1989 for an early pioneering work in finance and ZhangWB 1991 for a synergetically inspired view).

One idea this document tries to convey is the basic difference between standard methods of explanation, which are based on a global parameterization of a macroscopic dynamics and the "microscopic representation" or "agent-oriented" simulation method of explanation.

To illustrate the older methods, think of the high-school textbook problem of computing the trajectory of a stone under the influence of gravitation. The stone is treated as a point mass (that is the global parameterization), and we use Newton's Laws (that's the macroscopic dynamics).

This kind of explanation is not always useful. For instance in order to study the cracking and crumbling of the stone under pressure, one would have to consider its structure made of clusters of (of clusters ...) of smaller stone parts. In many physical, economic etc situations, we find it is most natural to view the system under study as a collection of "microscopic" similar units (or "agents"), which interact among themselves in some well-defined way and (co-)evolve in time.

Experience have shown that even a simple dynamics applied to a system consisting of many similar interacting agents may produce interesting macroscopic effects. The properties which arise from the collective behavior of many similar elements are called emergent properties.

Some examples of systems with emergent properties are:

Animal (and human) populations (cities/countries) are composed of individuals which are constantly born (and die), and compete with other members of their species (for food, mates etc).

The classical (scalar) Lotka-Volterra system:

$$w(t+1) = (1 + \text{birth} - \text{death}) * w(t) - \text{competition} * w(t)^2 \tag{1}$$

was invented specifically in order to model the time evolution of such populations $w(t)$. We will show that its straightforward multi-agent generalization eq. (2) which treats the population as a collection of sub-populations w_i, presents emergent properties very different from the eq. (1) which treats the entire population as a single variable (similarly to the stone/point mass example above).

The stock market is parametrized macroscopically by the index $w(t)$ which is proportional to the capitalization (total worth of the money invested) in the traded equities. For a fixed amount of shares, this is a measure of the price of the shares traded in the market. Models similar to the stone/point-mass/Newton law example above were introduced in the past (Baillie and Bollerslev 1990) and lead to models which treat the index/share price $w(t+1)$ as a single entity with time evolution governed by stochastic differential equations.

The microscopic representation method models the market index $\bar{w}(t)$ as an collective quantity emerging from the interactions of a macroscopic number of traders.

More precisely, the "microscopic representation" of the stock market is composed of many investors $i = 1, ..., N$ each having a certain personal wealth $w_i(t)$. The investors buy and sell stocks, using more or less complicated strategies designed to maximize their wealth. The resulting index is then proportional to the total sum of the individual investors wealth.

Another ("orthogonal") way to characterize the financial markets is to consider as elementary degrees of freedom the individual stocks (i.e. the capitalization $w_i(t)$ of the individual companies i traded in the market).

We are going to use the Generalized Lotka-Volterra (GLV) equation system

$$w_i(t+1) = \lambda(t) w_i(t) + a(t)\bar{w}(t) - c(t)\bar{w}(t) w_i(t) \quad , \quad i = 1, ..., N \tag{2}$$

to construct a very simple model of the stock market. In spite of being very simple, the model yields (and in doing so - explains) the Pareto distribution of wealth, an economical fact left for a long time (100 years!) without satisfactory explanation. In the case of the previous example of populations/cities/countries sizes, the system eq. (2) results from the assumption that the w_i individuals belonging to each city/country i interact similarly (in the stochastic sense) to all the individual members of the other cities/countries. More precisely, eq. (2) would result from the following assumptions:

- for each of the w_i citizen of a city, there is a λ probability that he will attract (or give birth to) a new citizen. One may include as a negative contribution to λ the probability for the citizen to die or leave the town without preferred destination.

- each of the w_i citizens of a city has a probability c/N to be attracted by one of the $N\bar{w}$ citizens of the existing cities and leave the town i.

- the citizens which left own town but are not bound to a particular town, have the probability a to end up in any of the i towns.

The analysis below shows that if the last term is negligible, the exponent α is likely to be 1.

Systems made of auto-catalytic (terms λ and a) competing (term c) elements are widespread in nature. Another example, very different from the previous: clouds are collections of water droplets, which grow by taking in water vapor from the surrounding air. Agent-oriented models may explain the observed distribution of cloud sizes and shapes without going into all the very complicated physical details.

To make things more concrete, we will often use in the following discussion terminology borrowed from the GLV model of the stock market. However, our results and insights will be quite generic and applicable to other systems.

For example, we will often use the term 'traders' i instead of microscopic elements and talk about the 'wealth' w_i of each trader and the "average wealth" $\bar{w}(t)$:

$$\bar{w}(t) = \frac{1}{N}\sum_i w_i(t) \tag{3}$$

All these names should be suitably renamed when used in other contexts.

III. PROBABILITY DISTRIBUTIONS AS PREDICTIVE OUTPUT OF MODELING

Modeling the stochastic evolution in time $t = 1, ..., T$ of a system of type eq. (1) produces a lot of raw data. It is not very useful to produce more and more sequences of numbers

$$w(t); \quad t = 1, ..., T$$

without a way to analyze this data.

A natural way is to collect a lot of such sequences and analyze them statistically. A way to look at it is to imagine a large set of (uncoupled) traders $w_i, \quad i = 1, ..., N$, subject to the same stochastic dynamics, and compute their distribution. More precisely, assuming each data sequence i produced by eq. (1) describes the evolution of one trader i

$$w_i(t); \quad t = 1, ..., T$$

we may compute over a large set of traders $i = 1, ..., N$ the number of traders $N_t(w)$ with wealth $w_i(t)$ in the interval $(w, w + dw)$. In the limit of infinite N this defines the individual wealth probability distribution

$$P_t(w)dw = N_t(w)/N \tag{4}$$

This way of looking at the problem is conductive to a model like eq. (2) in which the various traders **do** interact e.g. through terms including their average $\bar{w}$. As we will see, this is a key ingredient in solving some of the problems with the one-agent models of the type eq. (1) and of course a way to make the models more realistic.

IV. THE GENERAL SHAPE OF THE PROBABILITY DISTRIBUTIONS

It may be a little surprising, but in certain conditions, even for non-stationary interacting non-linear dynamical systems of the type eq. (2), we may predict (without performing any computer runs) how the wealth distributions $P_t(w)$ produced by our models will look like.

Before going into a more detailed analysis we list here some obvious properties which the probability distribution has to obey.

- The distribution vanishes for negative values.

$$P(w) = 0 \ for \ w < 0$$

  This property follows from the way we choose the initial wealth values, and the nature of the dynamics. In practice this corresponds to the fact that population sizes, capitalization of companies, cloud sizes, etc. assume only positive values.

- The value zero usually has vanishing probability distribution $P(0) = 0$. This follows from the continuity of the distribution, and the previous characteristic.

- The distribution has a tail which may obey a power-law

$$P(w) \sim w^{-1-\alpha} \tag{5}$$

  or not. In any case, distributions must decay to zero at infinity:

$$\lim_{w->\infty} P(w) = 0 \tag{6}$$

  otherwise the probability to have finite w_i values would vanish.

- The distribution has a maximum. A "nice" continuous and non-negative function will have a maximum between two zeroes. Here we have one at zero, and one at infinity.

The remaining non-trivial issues are:

- The behavior of the tail, in particular whether it decreases as a power-law eq. (5), log-normal, exponential or some other shape.
- The location and height of the maximum
- The behavior of the rising part
- The relation between the tail shape and the parameters of the model. We will see that in certain conditions, the tail behavior is quite universal in as far as it is quantitatively independent on most of the parameters and it is preserved even in generically non-stationary conditions.

In this article we will be interested mainly in the shape of the tail of the probability distribution $P(w)$, in particular whether it is a power-law eq. (5) or not.

The importance of the power laws is far from purely academic: the power laws are the bridges between the simple microscopic elementary laws acting at the individual level and the complex macroscopic phenomena acting at the collective level. They insure that the dynamics does cover many dynamical scales rather than acting only at the smallest and/or largest scales of the system.

Moreover, power laws insure that the dynamics of the intermediate scales of the system is largely independent on the microscopic details and the macroscopic external conditions constraining the system. Consequently the macroscopic complexity is not the direct and linear result of the microscopic details but rather a generic consequence of the self-organization taking place in systems with many interacting parts and inter-related feedback loops.

This means that one can hope to extract macroscopic and mesoscopic laws which hold for large classes of microscopic laws in an universal way.

This unifying and predictive power of the power laws made them the central objects in many branches of science starting with quantum field theory and statistical mechanics and ending with ecology and economics. In fact the necessity for such laws in fundamental physics lead to very "un-natural" (t'Hooft 1979) fine-tuning procedures designed to enforce

them.

One can say that the emergence of the power laws is a *sine qua non* condition for the emergence of the macroscopic world out of local microscopic elementary laws of nature. As such, it becomes a fundamental natural law on its own.

The present paper shows that power laws emerge generically in the most simple and natural models which were considered in the past in the modeling of chemical, biological and social systems.

V. MULTIPLICATIVE RANDOM DYNAMICS AND LOG-NORMAL DISTRIBUTIONS

Power-law probability distributions eq. (5) with exponent -1 (i.e. $\alpha = 0$), can be obtained analytically from a multiplicative process (Redner 1990; Shlesinger 1982):

$$w(t+1) = \lambda(t) w(t) \tag{7}$$

where the random variables $\lambda(t)$ are extracted from a fixed probability distribution $\Pi(\lambda)$ with positive support.

Indeed, in order to obtain the distribution of $w(t)$ in the large t limit, one takes the logarithm on both sides of eq. (7) and uses the notations $\mu = \ln \lambda$, $x = \ln w$. With these notations eq. (7) becomes:

$$x(t+1) = \mu(t) + x(t) \tag{8}$$

The respective probability distributions $\rho(\mu)$ and $\mathcal{P}(x)$ for μ and x are related to the distributions for λ and w by the identities:

$$\rho(\ln \lambda) d(\ln \lambda) = \Pi(\lambda) d\lambda \tag{9}$$

and respectively

$$\mathcal{P}(\ln w) d(\ln w) = P(w) dw \tag{10}$$

which mean:

$$\rho(\mu)d\mu = (\exp\mu)\Pi(\exp\mu)d\mu \tag{11}$$

and respectively

$$\mathcal{P}(x)dx = (\exp x)P(\exp x)dx \tag{12}$$

The interpretation of eq. (8) is that $x(t+1)$ is the sum of the constant $x(0) = \ln w(0)$, and t random variables $\mu(t) = \ln\lambda(t)$ extracted from the fixed distribution $\rho(\mu)$. Under the general conditions of the Central Limit Theorem (CLT) we get for $x(t)$ at large t the normal distribution:

$$\mathcal{P}_t(x) \sim \frac{1}{\sqrt{2\pi\sigma^2_{\ln\lambda}t}} \exp -\frac{(x- < x >)^2}{2\sigma^2_{\ln\lambda}t} \tag{13}$$

where

$$< x >= tv \equiv t < \ln\lambda > \tag{14}$$

and

$$\sigma^2_{\ln\lambda} =< (\ln\lambda)^2 > - < \ln\lambda >^2 \tag{15}$$

Note that the width of the normal distribution $\mathcal{P}_t(x)$ (the denominator of the expression in the exponential in eq. (13)) is

$$\sigma^2_x = t\sigma^2_{\ln\lambda} \tag{16}$$

and increases indefinitely with time. This means that the distribution $\mathcal{P}_t(x)$ becomes independent of x

$$\mathcal{P}_t(x) \sim \frac{1}{\sqrt{2\pi\sigma^2_{\ln\lambda}t}} \tag{17}$$

in an ever-increasing neighborhood of $< x >$ (where the exponential is close to 1).

Transforming eq. (17) back to the w variables by using eq. (10) one gets

$$P(w)dw = \frac{1}{\sqrt{2\pi\sigma_{\ln\lambda}^2 t}} d(\ln w) \tag{18}$$

i.e.

$$P(w)dw \sim 1/w \; dw \tag{19}$$

Graphically, this means that as the time goes to infinity, $P(w)$ is an ever "flattening" distribution approaching w^{-1} in an ever expanding neighborhood.

VI. NON-INTERACTIVE MULTIPLICATIVE PROCESSES WITH FIXED LOWER BOUND

In order to obtain a power-law probability distributions $P(w) \sim w^{-1-\alpha}$ with exponent $\alpha > 0$ the multiplicative process eq. (7) has to be modified (Yule 1924, Champernowne 1953, Simon and Bonini 1958, Ijiri and Simon 1977) in such a way that the variation of $w(t)$ under eq. (7) will be constrained by a lower bound (or barrier) w_{min}

$$w(t) > w_{min}. \tag{20}$$

In terms of $x(t) = \ln w(t)$, eq. (8) becomes supplemented by the lower bound:

$$x(t) > x_{min} \equiv \ln(w_{min}) \tag{21}$$

More specifically the dynamics of $w(t)$ consists at each time t in the updating eq. (7) (or - equivalently - eq. (8)) **except if** this results in $w(t+1) < w_{min}$ (or, equivalently $x(t+1) < x_{min} = \ln(w_{min})$) in which case the updated new value is $w(t+1) = w_{min}$ (respectively $x(t+1) = x_{min}$).

These modifications are obviously not allowed by the Central Limit Theorem, and consequently, the derivation in the previous section which led to the log-normal distribution eqs. (13)(19) cannot be applied.

Instead, one can obtain intuition on the modified system eq. (7)(20) by realizing that in fact the system eq. (8) with the constraint eq. (21) can be interpreted as the vertical motion

of a molecule in the earth gravitational field above the earth surface. In this case the "earth surface" is x_{min}, the gravitationally induced downward drift is $v =< x > /t =< \ln \lambda >$ and the diffusion per unit time is parametrized by the squared standard deviation $\sigma^2_{\ln \lambda}$.

For $v < 0$ this is the barometric problem and it has the static solution:

$$P(x) \sim \exp -x/kT \tag{22}$$

which when re-expressed in terms of w's becomes (cf. eq. (10)):

$$P(w)dw \sim e^{-(\ln w)/kT} d \ln w \tag{23}$$

i.e.

$$P(w) \sim w^{-1-1/kT} \tag{24}$$

In order to estimate the value of the exponent

$$\alpha = 1/kT \tag{25}$$

one can substitute the form eq. (24) in the master equation which governs the evolution of the probability distribution of the process eq. (7). The master equation expresses the flow of probability between the various values of w as $w(t)$ is updated to $w(t+1) = \lambda w(t)$. More precisely, the probability for $w(t+1) = \lambda w(t)$ to equal a certain value w is the integral over λ (weighted by $\Pi(\lambda)$) of the probabilities that $w(t) = w/\lambda$. This leads in equilibrium to the relation:

$$P(w) = \int \Pi(\lambda) P(w/\lambda) d(w/\lambda) \tag{26}$$

i.e., substituting (24)(25) into eq. (26):

$$w^{-1-\alpha} = \int \Pi(\lambda)(w/\lambda)^{-1-\alpha} d(w/\lambda) \tag{27}$$

or, by dividing both members by $w^{-1-\alpha}$:

$$1 = \int \lambda^{\alpha} \Pi(\lambda) d\lambda \tag{28}$$

This means that the value of α is given by the transcendental equation: (Solomon and Levy 1996):

$$< \lambda^\alpha >= 1 \tag{29}$$

One may wonder why this equation does not hold for the problem in the previous section (the system eq. (11) without the lower bound eq. (20)) since the lower bound w_{min} does not seem to appear in the formula eq. (29). The answer is that in the absence of the lower bound, there is no stationary equation (26) as the system evolves forever towards the non-normalizable solution $P(w) \sim w^{-1}$. In fact, formally, $\alpha = 0$ *is* a solution of eq. (29). This "solution" is in fact the relevant one in the case $< \ln \lambda > \geq 0$.

Another nontrivial fact is that the limit $w_{min} \to 0$ leads to $\alpha \to 1$ rather than $\alpha \to 0$ (which is the value in the total absence of a lower bound). This nonuniform behavior of the limits $N \to \infty$ and $w_{min} \to 0$ is related to the non-trivial thermodynamic limit of the system (Biham et al. 1998).

One of the problems with the solution to eq. (29) is that while it is independent on w_{min},it is highly dependent on the shape and position of the distribution $\Pi(\lambda)$ of the random factor $\lambda(t)$ and consequently it is highly dependent on the changes in the dynamics eq. (11).

In the systems introduced in the following sections, the situation will be the opposite: the characteristics of $\Pi(\lambda)$ will be largely irrelevant and the lower bound eq. (20) will play a central role.

In particular, the distribution of the social wealth $P(w)$ and the inflation $(d\bar{w}/dt)$ of the system will depend on the social security policy, i.e. on the poverty bound w_{min} below of which the individuals are subsidized (Anderson 1995). The relevant parameter is the ratio

$$q = w_{min}/\bar{w} \tag{30}$$

between the minimal wealth w_{min} and the average wealth:

$$\bar{w} = 1/N \sum_i w_i \tag{31}$$

In the presence of inflation, it is q rather then w_{min} which has to be fixed since a fixed minimal wealth independent on the changes in the average wealth would be not very effective for long time periods.

To prepare the formalism for those more sophisticated applications we will deduce below an identity relating the exponent $\alpha = 1/kT$ to the ratio q eq. (30).

Such a relation can be obtained using in eq. (24)(25) the fact that the total probability is 1:

$$\int_{w_{min}}^{\infty} P(w)dw = 1 \tag{32}$$

i.e.

$$Const. \int_{w_{min}}^{\infty} w^{-1-\alpha}dw = 1 \tag{33}$$

and the fact that the average of w is $\bar{w}$:

$$\int_{w_{min}}^{\infty} wP(w)dw = \bar{w} \tag{34}$$

i.e

$$Const. \int_{w_{min}}^{\infty} w^{-\alpha}dw = \bar{w} \tag{35}$$

By extracting $Const$ from eq. (33):

$$Const = \alpha w_{min}^{\alpha} \tag{36}$$

and introducing this value in eq. (35) one gets the relation:

$$\alpha w_{min}^{\alpha}[-w_{min}^{1-\alpha}/(1-\alpha)] = \bar{w} \tag{37}$$

By dividing both members by $\bar{w}\frac{\alpha}{1-\alpha}$ one obtains:

$$-w_{min}/\bar{w} = 1/\alpha - 1 \tag{38}$$

Considering the definition eq. (30) this reads:

$$\alpha = 1/(1-q) \tag{39}$$

This is a quite promising result because for the natural range of the lower bound ratio $0 < q < 1/2$ it predicts $1 < \alpha < 2$ which is the range of exponents observed in nature (which is far from the log-normal value $\alpha = 0$).

The main problem with the power-low generating mechanism based on the single-agent dynamics eq. (11)(20) is that it works only for negative values of $< \ln \lambda >$ (the barometric equation does not hold for a gravitational field directed upwards).

This means that (except for the neighborhood of w_{min}) $w(t+1)$ is typically smaller than $w(t)$. This is not the case in nature where populations, economies are expanding (at least for certain time periods).

Moreover, the exponent α of the power law is highly unstable to fluctuations in the parameters of the system. In particular, trying to model large changes in $\bar{w}$ one is lead to large fluctuations of q and consequently (cf. eq. (39)) to large variations in the exponent α. This again is in disagreement with the extreme stability of the exponents α observed in nature.

We will see later how the multi-agent GLV solves these problems. The key feature is to introduce interactions between the individual elements w_i through the inclusion of terms (or lower bounds) proportional to $\bar{w}$.

VII. MULTIPLICATIVE PROCESSES COUPLED THROUGH THE LOWER BOUND

In the previous section we showed that power-laws eq (24) can be obtained from multiplicative stochastic dynamics with lower bound in the same way that the exponential laws eq. (22) can be obtained in additive stochastic dynamics eq. (8) bounded from below eq. (21).

The problematic points were: the instability of the exponent α to variations in the average wealth/population and the fact that the mechanism accounts only for "deflat-

ing"/"shrinking" of w.

However, the solution of these difficulties is already apparent in eq. (39): we consider a system of N degrees of freedom w_i (with $i = 1, ...N$) which is governed by the following dynamics:

at each time t, one of the w_i's is chosen randomly to be updated according to the formula:

$$w_i(t+1) = \lambda(t) w_i(t) \tag{40}$$

while all the other w_i's are left unchanged. At each instance, the random factor λ is extracted anew from an i-independent probability distribution $\Pi(\lambda)$.

The only exception to the prescription eq. (40) is if following the updating eq. (40) $w_i(t+1)$ (or other w_k's) end up less then a certain fixed fraction $0 < q < 1$ of the average $\bar{w}$:

$$w_i(t+1) < q\bar{w}(t) \tag{41}$$

The prescription, if eq. (41) happens, is to further update the affected w_j's to:

$$w_j(t+1) = q\bar{w}(t). \tag{42}$$

The wonderful property of the system eq. (40–42) is that when re-expressed in terms of the variables

$$v_i(t) = w_i(t)/\bar{w}(t) \tag{43}$$

it leads to a system very close to the system eq. (7),(20):

$$v_i(t+1) = \tilde{\lambda}(t) v_i(t) \tag{44}$$

$$v_i(t+1) > q \tag{45}$$

where the effective multiplicative factor in eq. (44) is:

$$\tilde{\lambda}(t) = \lambda(t)\bar{w}(t)/\bar{w}(t+1) \tag{46}$$

Note that this solves the problem of $< \ln \lambda >> 0$ because the multiplication with the ("renormalization") factor $\bar{w}(t)/\bar{w}(t+1)$ takes care that the v distribution is never running to infinity (in fact it insures that $\bar{v}(t) = 1$ always).

Another way to see how this solves the problems with the single "particle" dynamics eqs. (7)(20) is to realize that now, with a $q\bar{w}$ lower bound, even if the average $\bar{w}(t)$ runs to infinity, the lower bound runs after it in such a way as to insure a stationary value of α.

Indeed, according to eq. (24), the v_i dynamics eq. (44–45) leads to a v_i distribution

$$P(v) \sim v^{-1-\alpha} \tag{47}$$

which in turn implies

$$P(w) \sim w^{-1-\alpha} \tag{48}$$

Since we do not have a close analytic formula for $\tilde{\lambda}$ eq. (46) in terms of the model parameters, the transcendental equation eq. (29) is not useful here.

However, since $\bar{v} = 1$ by definition, one gets according eq. (39) the following close formula for the exponent α:

$$\alpha = 1/(1-q) \tag{49}$$

This means that even for significantly time-varying distributions $\Pi_t(\lambda)$ in eq. (40), the exponent of the power law remains time-invariant.

The above results are quite non-trivial in as far as they predict the stochastic behavior of highly interactive and time non-stationary systems eqs. (40)-(42) by relating them formally to non-interacting static statistical systems eqs. (7)(20).

We will continue this line in the following sections an reduce the highly nontrivial GLV system to a rather simple single-agent linear stochastic equation.

VIII. THE SINGLE-AGENT LINEAR STOCHASTIC EQUATION

The discrete equation below is a more elaborate version of the single-agent model introduced in Section 6. Here the lower bound is supplied effectively by an additive random term

ρ, instead of an explicit barrier eq. (20):

$$w(t+1) = \lambda(t)w(t) + \rho(t) \tag{50}$$

The equation (50) may (crudely) describe the time-evolution of wealth for one trader in the stock market. In this context w may represent the wealth of one trader, who performs at each time-step a stock transaction. $w(t)$ is the wealth at time t, and $w(t+1)$ is the wealth at time $t+1$.

λ and ρ are random variables extracted from positive probability distributions. λ may be called the "success factor", and ρ is a "restocking term", which takes into account the wealth acquired from external sources (e.g. using the state built infra-structure, subsidies etc).

The dynamics eq. (50) leads to a power-law distribution $P(w) \sim w^{-1-\alpha}$, if $< \ln \lambda >< 0$. More precisely, the $P(w)$ has a power-tail for values of w for which $\rho(t)$ is negligible with respect to $\lambda(t)w(t)$. If $< \ln \lambda > \geq 0$ the distribution is a log-normal expanding in time, which in the infinite time limit corresponds to a power-law with exponent $-1-\alpha = -1$. I.e. the mechanism eq. (50) does not explain expanding economies with $\alpha > 0$.

Since the distributions (and initial values of w) are positive, the $\rho(t)$ term keeps the value of $w(t)$ above a certain minimal value of order $\bar{\rho}$. Therefore, for large enough values of w the dynamics is indistinguishable from our previous model, the multiplicative process (7) with fixed barrier eq. (20).

It is therefore not surprising that it can be rigorously proven (Kesten 1973) that eq. (50) leads to a power law eq. (24) with exponent α given by the transcendental equation (29):

$$< \lambda^{\alpha} >= 1 \tag{51}$$

It is again notable that (as in the case of the w_{min} lower bound), the exponent α is totally independent on the distribution of the additive term ρ while it is highly sensitive to the shape and position of the $\Pi(\lambda)$ distribution. We will see that in GLV the situation is reversed.

IX. THE GENERALIZED LOTKA-VOLTERRA SYSTEM

The GLV (Solomon and Levy 96) implements a more complex dynamics than the eq. (50). The advantages however overweight the difficulties:

- The GLV model insures a stable exponent α of the power-law $P(w) \sim w^{-1-\alpha}$ even in the presence of large fluctuations of the parameters.

- The value of $< \lambda >$ and the average $\bar{w}$ can vary during the run (and between the runs) without affecting the exponent α of the power-law distribution.

- λ can take typical values both larger or smaller than 1.

The GLV is an interactive multi-agent model. We have N traders, each having wealth w_i, and each w_i is evolving in time according to:

$$w_i(t+1) = \lambda(t) w_i(t) + a(t)\bar{w}(t) - c(t)\bar{w}(t) w_i(t) \tag{52}$$

Here $\bar{w}$ is the average wealth, which supplies the coupling between the traders:

$$\bar{w} = (w_1 + w_2 + \ldots + w_N)/N \tag{53}$$

λ is a positive random variable with a probability distribution $\Pi(\lambda)$ similar to the λ success factor we used in eq. (50) in the previous section. The dramatic difference is that now, λ can take systematically values larger than 1 and in fact its distribution can vary in time (and have time intervals with $< \ln \lambda >$ both smaller and larger than 0).

The coefficients a and c are in general functions of time, reflecting the changing conditions in the environment.

The coefficient a expresses the auto-catalytic property of wealth at the social level, i.e. it represents the wealth the individuals receive as members of the society in subsidies, services and social benefits. That is the reason it is proportional to the average wealth.

The coefficient c originates in the competition between each individual and the rest of society. It has the effect of limiting the growth of $\bar{w}$ to values sustainable for the current conditions and resources.

X. REDUCING GLV TO A SET OF INDEPENDENT EQUATIONS (50)

Summing the GLV eq. (52) over i, and taking the local time average one gets for the local time average $<\bar{w}>(t)$ an equation similar to the scalar LV equation eq. (1).

$$N<\bar{w}>=<\lambda>N<\bar{w}>+<a(t)>N<\bar{w}>-<c(t)>N<\bar{w}>^2 \tag{54}$$

which gives :

$$<\bar{w}>=\frac{-1+<\lambda>+<a(t)>}{<c(t)>} \tag{55}$$

Neglecting the fluctuations of the average $<\bar{w}>(t)$ during the updating of a single individual i, one can substitute $<\bar{w}>(t)$ for $\bar{w}$ it in the last term of eq. (52) and regroup the terms linear in w_i:

$$w_i(t+1)=[1+\lambda(t)-<\lambda(t)>-<a(t)>]w_i(t)+a(t)\bar{w}(t) \tag{56}$$

Introducing like we did in eq. (43) wealth values normalized by the average wealth:

$$v_i(t)=w_i(t)/\bar{w}(t) \tag{57}$$

the equation (56) becomes:

$$v_i(t+1)=[1+\lambda(t)-<\lambda(t)>-<a(t)>)]v_i(t)+a(t) \tag{58}$$

where we neglected again the fluctuation of $\bar{w}$ during the time $t \rightarrow t+1$ and put $\frac{\bar{w}(t)}{\bar{w}(t+1)}=1$. This can be justified rigorously for $\alpha>1$ in the large N limit because then the size of the largest w_i is (cf. Solomon 1998) of order $O(\bar{w}N^{1-\alpha})<<1$ and therefore the changes in $\bar{w}(t)$ induced by any w_i updating are negligible of the leading order. In finite systems $N<\infty$ and for $\alpha<1$, there are (computable) corrections.

The key observation now is that the system eq. (58) has the form of N decoupled equations of the form (50) with the effective multiplicative stochastic factor

$$\tilde{\lambda}=1+\lambda(t)-<\lambda(t)>-<a(t)> \tag{59}$$

This means that the results of the single linear stochastic agent model eq. (50) can be applied now to the $v_i(t)$'s in order to obtain

$$P(v) \sim v^{-1-\alpha} \tag{60}$$

which in turn implies

$$P(w) \sim w^{-1-\alpha} \tag{61}$$

with the exponent α dictated by the equation eq. (51)(59):

$$< [1 + \lambda(t) - < \lambda(t) > -a(t)]^{\alpha} >= 1 \tag{62}$$

This shows that the wealth distribution in a economic model based on GLV is a power law. In addition, it is easy to see using (58) that the elementary steps in the time variations of the average wealth are distributed by a (truncated) power law. Accordingly, it was predicted that the market returns will be distributed by $L_{\alpha}(\bar{w})$ a truncated Levy distribution of index α (Solomon 1998). This turned out to be in accordance with the actual experimental data (Mantegna and Stanley 1996).

Note that similarly to the passage from the single agent system eq. (7)(20) to the many agents system coupled by the lower bound eq. (40)(41), here too, the **formal** reduction of the GLV to a single agent system implies very different properties in the actual "physical" system.

In particular while in the single agent system the average of λ was crucial in the fixing of the exponent α and in fact λ had to have an average less then 1, in the multi agent model, the average of λ cancels in the expression (59) for $\tilde{\lambda}$. On the other hand, while the additive term in eq (50) had no role in the fixing of α, the corresponding term a is one of the crucial factors determining α in GLV (cf. eq. (62)). Moreover, while the parameters in the eq. (50) model allowed time variations in the $\bar{w}$ only at the price of variations in the exponent α, in the GLV system, one can arbitrarily change the ecological/economic conditions c so as to vary the total population/wealth by orders of magnitude (cf. eq. (55)) without affecting the power law and its exponent ($\bar{\lambda}$ is independent on c and so is the solution of $< \tilde{\lambda}^{\alpha} >= 1$.

XI. THE FINANCIAL INTERPRETATION OF GLV

In this section we discuss the various terms appearing in the equations, their interpretation in the financial markets applications, their effects and their implications for the financial markets phenomenology.

We first discuss the assumption that the individual investments/gains/losses are proportional to the individual wealth:

$$w(t+1) = \lambda w(t)$$

This is actually not true for the low income/wealth individuals whom incomes do not originate in the stock market. In fact the additive term $a\bar{w}$ tries to account for the departures related to additional amounts originating in subsidies, salaries and other fixed incomes. However, for the range of wealth where one expects power laws to hold ($w > \bar{w}$) it is well documented that the investment policies, the investment decisions and the measured yearly income are in fact proportional to the wealth itself.

The statistic uniformity of the relative gains and losses of the market participants is a weak form to express the fairness of the market and the lack arbitrage opportunities (opportunities to obtain systematically higher gains $\lambda - 1$ than the market average without assuming higher risks): for instance, if the distribution of λ would be systematically larger for small-w-investors, then the large-w-investors would only have to split their wealth in independently managed parts to mimic that low-w superior performance. This would lead to an equalization of their λ to the λ of the low-w investors. Therefore in the end, the distribution $\Pi(\lambda)$ will end up w-independent as we assumed it to be in GLV from the beginning.

Almost every realistic microscopic market model we have studied in the past shares this characteristics of w-independent $\Pi(\lambda)$ distribution.

Turning to the terms relevant for the lower-bound w region, the assumption that the average wealth contributes to the individual wealth $a\bar{w}$ and the alternative mechanism assuming a lower bound proportional to the average $q\bar{w}$ are both simplified mechanisms to

prevent the wealth to decrease indefinitely.

In practice they might be objectionable: it is not very clear that the state subsidies can be invoked to save bankrupt investors (though this happens often when large important employers are at risk or when their collapse would endanger the stability of the entire system). In any case the term $a\bar{w}$ is appropriately taking into account the arbitrariness of the money denominations. More precisely, assume that (by inflation or by currency renaming) the nominal value of all the money in the economy becomes 10 times larger. Then the subsidies term $a\bar{w}$ will become 10 times larger preserving in this way the actual absolute value. So to speak, multiplying a quantity by $\bar{w}$ expresses it in "absolute currency".

These mechanisms controlling the lower bound behavior of the system may be considered as just parametrizations of the continuum flow of investors/capital to and from the large-w investor ranges relevant for the financial markets trading. It is however possible that the subsidies to the very poorest tumble their way (through the multiplicative random walk) into the middle classes and end-up in the large-w scaling region in the way our models suggest.

More theoretical and experimental research is necessary in order to discriminate between the various alternatives (or in order to recognize them as facets of the same phenomenon).

For instance one can look at the relation between q and α as basically kinematic in the sense that given the power law, it is unavoidable that the lower bound $q\bar{w}$ would govern the exponent of the power law.

It would still be interesting to study in detail how the wealth pumped at the lower-bound barrier makes its way to the large w tail and or, alternatively in the case of stationary $\bar{w}$ to find the way in which the additive subsidies to the low-w individuals in our models are covered by the "middle classes".

For instance in the $q\bar{w}$ lower-bound model, the dominant mechanism of extracting wealth from the middle class seems to be the accelerated inflation: the updating of the individual w_i's induces changes in $\bar{w}$ which in turn induces changes in the position of the lower bound $q\bar{w}$ which leads to the necessity to subsidize immediately all the w_i's situated between the old and the new poverty line. This in turn leads to increase in the $\bar{w}$ and to the completion

of the positive feedback loop. When looking at the normalized wealth $v_i = w_i/\bar{w}$, this accelerated inflation is effectively a proportional tax which reduces the λ gains to smaller relative gains $\tilde{\lambda} = \lambda w(t)/w(t+1)$.

In the case of the GLV model, there is no inflation (for constant c, $\bar{w}$ is - modulo local fluctuations - constant in time). The extraction from the middle class of the money for subsidies is quite explicit from the way the subsidies term $a\bar{w}$ is affecting negatively $(-a)$ the $\tilde{\lambda}$ gain value eq. (39). However most of the loss in $\tilde{\lambda}$ eq. (39) is the term $(- < \lambda >)$ and is due mainly to the competition between the large traders $-cw_i\bar{w}$. This term is in fact equivalent with enforcing a proportional tax. The best way to see it is to recall from above analysis that multiplying the wealth-proportional quantity cw_i by $\bar{w}$ expresses it in "absolute currency".

It would appear that the government has the choice of enforcing such a proportional tax and keep $\bar{w}$ roughly constant, or just print the money which it dispenses to the poor and let the inflation tax the other classes. One can of course make compromises between these 2 extremes by allowing both taxation and inflation. In any case, the net result can be only variations in the relative size of the middle class (variations of α) as the generic emergence of the power law will be very difficult to avoid.

A government may try to get a more equalitarian distributions (increasing α) by increasing q. However this would bring an even larger population in the neighborhood of the poverty line. This would require more and more frequent subsidies to enforce and consequently (according to the analysis above) larger taxes and/or faster inflation.

Fearing this, another government might decide for low values of α. This would not only mean an increase of the ratio between the richest and poorest which might be morally questionable but also lead to dramatically unstable fluctuations in the market (e.g. in $\bar{w}$). Indeed, one can show that typically the largest trader owns $O(N^{1-\alpha})$ of the total wealth. For finite N and very low q, α may be shown to drop below 1 (contrary to eq. (49)). This would imply that almost all the wealth is owned by the largest w_i. It is well know however that the single agent discrete logistic map eq. (1) leads generically to chaotic unstable dynamical

regime (May 1976).

In between these extremes it might be that there is no much choice of dynamically consistent α values outside the experimentally measured range $1.4 < \alpha < 1.7$. These values correspond (through eq. (49)) to poverty lines between $0.3 < q < 0.4$. Larger values of q would imply that almost everybody is subsidized while a significantly lower value would mean the poorest cannot literally live: the average wages in an economy are automatically tuned as to insure that a family of 3 can fulfill the needs considered basic by the society on a one-salary income. Somebody earning less than $\frac{1}{3}\bar{w}$ will therefore have serious problems to live a normal life. In fact, such a person might be exposed to hunger (even in very rich economies) since even the prices of the basic food are tuned to the level of affordability of the average family in the given economy.

Far from implying a fatalistic attitude of the economic facts of life, the analysis of our simplified models might lead in more realistic instances to ideas and prescriptions for dynamically steering the social economic policies to optimal parameters both from the technical/efficiency and from the human/moral point of view.

They might help transcend (through unification) the traditional dichotomy by which science is only in charge with deciding true from false while humanities are in charge only with discerning good from bad.

XII. FURTHER ECONOMICS APPLICATIONS OF GLV

As mentioned a few times, the GLV systems can be applied to many power laws.

Even within the financial framework there are a few apparently different realizations of the GLV dynamics and of the power laws.

For instance, one can consider the market as a set of companies $i = 1, ..., N$ whose shares are traded and whose prices vary in time accordingly.

One can interpret then w_i as the capitalization of the company i i.e. the total wealth of all the market shares of the company.

The time evolution of w_i can still be represented by eq. (2). In this case, λ represents the fluctuations in the market worth of the company. For a given total number of shares λ is measured by the change in the individual share price. These changes take place during individual transactions and are typically fractions of the nominal share price (measured in percents or in points).

With such an interpretation, a represents the correlation between the worth of each company and the market index. This correlation is similar for entire classes of shares and differences in the a's of various economic sectors result in the modern portfolio theory in "risk premia" which affect their effective returns $\tilde{\lambda}$ in a way similar to our formulae in a previous section (low correlation a corresponds to larger effective incomes).

c represents the competition between the companies for the finite amount of money in the market (and express also the limits in their own absolute worth). We do not need to consider c a constant. Time increases in the resources may lead to lower values of c which in turn lead to increases in $\bar{w}$. However, as seen before, such changes do not affect the exponent of the power law distribution.

In this interpretation, the GLV model will predict the emergence of a power-law in the probability distribution of company sizes (capitalization). In particular, this would imply that the weights of the various companies composing the S&P 500 are distributed by a power law. In turn this would imply that the S&P fluctuations follow a truncated Levy distribution of corresponding index.

Yet another interpretation is to consider w_i as the size of coordinated trader sets (i.e. the number of traders adopting a similar investment policy) and assuming that the sizes of these sets vary self-catalytically according to the random factor λ while the a term represents the diffusion of traders between the sets. Such an autocatalytic dynamics for the trading schools is not surprising as their decision processes involve elements similar to the use of common language, common values, which are central in the dynamics of languages and nations (which fulfill power laws) too. The nonlinear term c represents then the competition between these investing schools for individual traders membership. In fact if a and c fulfill

the relation $a/c = \bar{w}$, the corresponding two terms in GLV taken together represent (in "absolute currency" $\bar{w}$) the act of each of the schools loosing (shedding) a certain c fraction of its followers which are then spread uniformly between the schools.

Such an interpretation of the GLV would predict a power-law in the distribution of trader schools sizes (similar to countries sizes). Again, such a distribution would account for the truncated Levy distribution of market fluctuations (with index equal to the exponent of the power-law governing trader sets sizes).

It would be interesting to discriminate between the validity of the various GLV interpretations by studying experimentally the market fluctuations and the distributions of respectively individual wealth, companies capitalization and correlated investors sets. It is not ruled out that some of these interpretations may be consistent one with the other. This would be consistent in turn with the modern portfolio theory claim on the existence of a "market portfolio" (stochastically) common to most of the traders.

Further developments of the GLV model will include introducing variable number of agents N, studying the role of the discretization of the w_i changes (due to indivisible units like people, shares, etc), taking into account the influence of the history of $w(t)$ on λ (like in the case of investing strategies, crashes memory, etc.).

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